LeanForControl

Anand Gokhale Francesco Bullo
University of California, Santa Barbara

September 2, 2026

Chapter 1

Linear systems

1.1 Observability and controllability matrices

For a finite-dimensional linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx,$$

with state matrix A, input matrix B, and output matrix C, this section collects the (finite-horizon) controllability and observability matrices and the bridge theorem connecting the textbook observability predicate to the kernel of the observability matrix.

1.1.1 Observability matrix

Definition 1. The *observability matrix* of a pair (A, C) with $A \in \mathbb{F}^{n\times n}$ and $C \in \mathbb{F}^{p\times n}$ is the block-row matrix

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ C\,A \\ C\,A^2 \\ \vdots \\ C\,A^{n-1} \end{bmatrix} \in \mathbb{F}^{(np)\times n}.$$

Rows are indexed by $\mathrm{Fin}\, n \times \mathrm{Fin}\, p$, so that A^k is available without casting $k : \mathrm{Fin}\, n$ through Fin.val.

Definition 2. A linear system (A, C) is *observable* when the only state $x \in \mathbb{F}^n$ for which

$$C\,A^k\,x = 0 \qquad \text{for every } k = 0, 1, \dots, n-1$$

is the zero state. This phrasing does not mention $\mathcal{O}(A, C)$, so the bridge theorem 8 has real content.

Theorem 3. *Block-row entry shape: at row (k, i) and column j, the observability matrix coincides with the (i, j) entry of $C\,A^k$:*

$$\mathcal{O}(A, C)_{(k,i),\,j} = (C\,A^k)_{i,\,j}.$$

Proof. Holds definitionally by the encoding chosen for theorem 1. ∎

Theorem 4. *For every state $x \in \mathbb{F}^n$ and every $(k, i) \in \mathrm{Fin}\, n \times \mathrm{Fin}\, p$,*

$$\left(\mathcal{O}(A, C) \cdot x\right)_{(k,i)} = \left(C\, A^k \cdot x\right)_i.$$

This is the definitional bridge that powers theorem 8.

Proof. Holds by definitional unfolding of $\mathcal{O}(A, C)$ (see theorem 1). □

1.1.2 Controllability matrix

Definition 5. The *controllability matrix* of a pair (A, B) with $A \in \mathbb{F}^{n\times n}$ and $B \in \mathbb{F}^{n\times m}$ is the block-column matrix

$$\mathcal{C}(A, B) = \begin{bmatrix} B & A\,B & A^2\,B & \cdots & A^{n-1}\,B \end{bmatrix} \in \mathbb{F}^{n\times(nm)}.$$

Columns are indexed by $\mathrm{Fin}\, n \times \mathrm{Fin}\, m$, so that A^k is available without casting $k : \mathrm{Fin}\, n$ through Fin.val.

Theorem 6. *Block-column entry shape: at row i and column (k, j), the controllability matrix coincides with the (i, j) entry of $A^k\, B$:*

$$\mathcal{C}(A, B)_{i,\,(k,j)} = (A^k\, B)_{i,\,j}.$$

Proof. Holds definitionally by the encoding chosen for theorem 5. □

Definition 7. A linear system (A, B) is *controllable* when every target state $x \in \mathbb{F}^n$ is reachable from the origin in n steps: there exist input vectors $u_0, u_1, \ldots, u_{n-1} \in \mathbb{F}^m$ such that

$$x = \sum_{k=0}^{n-1} A^k\, B\, u_k.$$

This phrasing does not name the controllability matrix, so the bridge theorem 10 has real content.

1.1.3 Milestone: observability iff trivial kernel of $\mathcal{O}$

Theorem 8. *A finite-dimensional linear system (A, C) is observable in the sense of theorem 2 if and only if the observability matrix $\mathcal{O}(A, C)$ has trivial kernel under matrix-vector multiplication:*

$$\mathrm{IsObservable}(A, C) \iff \left(\forall x,\ \mathcal{O}(A, C) \cdot x = 0 \Rightarrow x = 0\right).$$

Proof. Both directions follow from the bridge theorem 4, which identifies the (k, i)-coordinate of $\mathcal{O}(A, C) \cdot x$ with the i-coordinate of $(C\, A^k) \cdot x$. The forward direction reads the per-power kernel out of the assembled kernel coordinatewise; the reverse direction packs them back. □

1.1.4 Rank characterizations

Theorem 9. *A finite-dimensional system (A, C) is observable if and only if the observability matrix $\mathcal{O}(A, C)$ has full column rank, i.e.*

$$\mathrm{rank}\, \mathcal{O}(A, C) = n,$$

where n is the state dimension.

Proof. Chain the kernel-form milestone theorem 8 with the matrix-level bridge $(\forall x,\ M \cdot x = 0 \Rightarrow x = 0) \iff \operatorname{rank} M = n$ from 'MatrixLemmas', applied with $M = \mathcal{O}(A, C)$. $\square$

Theorem 10. *A linear system (A, B) is controllable if and only if the controllability matrix has full row rank,*

$$\operatorname{rank} \mathcal{C}(A, B) = n,$$

where n is the state dimension.

Proof. Combine the reindexing identity $\mathcal{C}(A, B) \cdot u = \sum_k A^k B \cdot u_k$ (with $u_k(j) = u(k, j)$) with the matrix-level bridge $(\forall y,\ \exists x,\ M\,x = y) \iff \operatorname{rank} M = n$ from 'MatrixLemmas', applied with $M = \mathcal{C}(A, B)$. $\square$

1.1.5 Hautus observability

Definition 11. The *unobservable subspace* of (A, C) is the A-invariant subspace

$$\mathcal{N}(A, C) \;=\; \{\, v \in \mathbb{C}^n \;:\; C\,A^k\,v = 0 \text{ for every } k = 0, 1, \ldots, n-1 \,\}.$$

Equivalently, $\mathcal{N}(A, C) = \ker \mathcal{O}(A, C)$, but here it is phrased without naming $\mathcal{O}$ so the bridge theorem 12 reads as content.

Theorem 12. *A finite-dimensional system (A, C) is observable in the sense of theorem 2 if and only if its unobservable subspace is trivial:*

$$\mathcal{N}(A, C) = \{0\} \iff \text{IsObservable}(A, C).$$

Proof. Both directions are membership unfoldings of theorem 11 against theorem 2. $\square$

Theorem 13. *The unobservable subspace is closed under the action of A: for every $v \in \mathcal{N}(A, C)$, also $A\,v \in \mathcal{N}(A, C)$.*

Proof. For $k = 0, \ldots, n-2$ this is direct from the definition. For $k = n-1$ we land at $C\,A^n\,v$, which Cayley-Hamilton rewrites as a $\mathbb{C}$-linear combination of $C\,A^i\,v$ for $i < n$. Each of those is zero, so the combination is zero. $\square$

Definition 14. The *Hautus observability matrix* of (A, C) at a complex number μ is the block-row matrix

$$H_{A,C}(\mu) \;=\; \begin{bmatrix} \mu I - A \\ C \end{bmatrix} \in \mathbb{C}^{(n+p)\times n}.$$

Theorem 15. *For every $\mu \in \mathbb{C}$ and $v \in \mathbb{C}^n$,*

$$H_{A,C}(\mu) \cdot v = 0 \iff (A\,v = \mu\,v \;\wedge\; C\,v = 0).$$

Proof. $\square$

Theorem 16. *If (A, C) is not observable, then there exists $\mu \in \mathbb{C}$ and a nonzero vector $v \in \mathbb{C}^n$ such that*

$$H_{A,C}(\mu) \cdot v = 0,$$

i.e. μ is an eigenvalue of A with an eigenvector in $\ker C$.

Proof. The unobservable subspace theorem 11 is nontrivial by theorem 12. It is A-invariant by theorem 13, hence finite-dimensional and invariant; over $\mathbb{C}$ this guarantees a nonzero eigenvector inside it. That vector is automatically in $\ker C$ (membership at $k = 0$), so it lies in the kernel of $H_{A,C}(\mu)$ via theorem 15. □

Theorem 17. *If $\mu \in \mathbb{C}$ and $v \neq 0$ satisfy $H_{A,C}(\mu) \cdot v = 0$, then (A, C) is not observable.*

Proof. The witness gives $Av = \mu v$ and $Cv = 0$. By induction on k, $A^k v = \mu^k v$, so $C\,A^k\,v = \mu^k\,Cv = 0$ for every k. The vector v is therefore in the unobservable subspace and is nonzero, so observability fails. □

Theorem 18. *A finite-dimensional system (A, C) over $\mathbb{C}$ is observable if and only if for every $\mu \in \mathbb{C}$ the Hautus matrix $H_{A,C}(\mu)$ has trivial kernel:*

$$\mathrm{IsObservable}(A, C) \iff \forall \mu \in \mathbb{C},\ \ker H_{A,C}(\mu) = \{0\}.$$

Proof. Combine theorem 16 (failure direction) and theorem 17 (converse). □

1.1.6 Hautus controllability

Definition 19. The *Hautus controllability matrix* of (A, B) at a complex number μ is the block-column matrix

$$H^{\mathrm{ctrl}}_{A,B}(\mu) \;=\; \begin{bmatrix} \mu I - A & B \end{bmatrix} \in \mathbb{C}^{n\times(n+m)}.$$

Theorem 20. *A finite-dimensional system (A, B) over $\mathbb{C}$ is controllable if and only if for every $\mu \in \mathbb{C}$ the Hautus matrix $H^{\mathrm{ctrl}}_{A,B}(\mu)$ has full row rank:*

$$\mathrm{IsControllable}(A, B) \iff \forall \mu \in \mathbb{C},\ \mathrm{rank}\, H^{\mathrm{ctrl}}_{A,B}(\mu) = n.$$

Proof. Combine the duality bridge theorem 20 *nope, that's the iff itself*; actually combine *the duality bridge* `isControllable_iff_isObservable_transpose` with the observability Hautus iff theorem 18, then convert the kernel-form RHS to the rank form via the matrix bridge `mulVec_kernel_trivial_iff_rank_eq_`
from ‘MatrixLemmas‘, and finally identify the transposed Hautus matrices. □

Chapter 2

Ordinary differential equations

2.1 Dini derivatives

Definition 21. The *upper right Dini derivative* of $f : \mathbb{R} \to \mathbb{R}$ at t is

$$D^+ f(t) \;=\; \limsup_{h \to 0^+} \frac{f(t+h) - f(t)}{h}.$$

When f is differentiable at t this equals $f'(t)$.

2.2 Gronwall–Bellman inequality

Theorem 22. ***Gronwall–Bellman inequality.*** *Let $\Lambda, \mu : [a,b] \to \mathbb{R}$ be continuous with $\mu \geq 0$, and let $y : [a,b] \to \mathbb{R}$ be continuous satisfying*

$$y(t) \;\leq\; \Lambda(t) + \int_a^t \mu(s)\, y(s)\, \mathrm{d}s \qquad \forall\, t \in [a,b].$$

Then

$$y(t) \;\leq\; \Lambda(t) + \int_a^t \Lambda(s)\, \mu(s)\, e^{\int_s^t \mu(\tau)\, \mathrm{d}\tau}\, \mathrm{d}s \qquad \forall\, t \in [a,b].$$

Proof. □

2.3 Continuous dependence on parameters

Definition 23. A function $x : [t_0, t_1] \to E$ is an *integral solution* of $\dot{x} = F(t, x)$ with initial value x_0 when

$$x(t) \;=\; x_0 + \int_{t_0}^t F(s,\, x(s))\, \mathrm{d}s \qquad \forall\, t \in [t_0, t_1].$$

Theorem 24. ***Theorem 3.5*** *(Continuous dependence on parameters). If y solves $\dot{y} = f(t, y)$ and z solves $\dot{z} = f(t, z) + g(t, z)$ with $\|g(t, x)\| \leq \alpha$ and $\|z_0 - y_0\| \leq \alpha$, and $\alpha(1 + 1/L)e^{L(t_1 - t_0)} \leq \varepsilon$, then $\|y(t) - z(t)\| \leq \varepsilon$ for all $t \in [t_0, t_1]$.*

Proof. Reduce to theorem 22 applied to $\|y - z\|$, using the L-Lipschitz bound on f and the α-bound on g. ☐

2.4 Comparison lemma

Theorem 25. ***Comparison Lemma 3.4.*** *If u solves $\dot{u} = f(t, u)$ with $u(t_0) = u_0$, and v is continuous with $D^+v(t) \leq f(t, v(t))$ and $v(t_0) \leq u_0$, then $v(t) \leq u(t)$ for all $t \in [t_0, t_1]$.*

Proof. For each $\lambda > 0$, the perturbed solution z_λ of $\dot{z} = f(t, z) + \lambda$ satisfies $v \leq z_\lambda$ by **??**. By theorem 24, $\|u - z_\lambda\| \leq \varepsilon/2$. Since $\varepsilon > 0$ is arbitrary, $v(t) \leq u(t)$. ☐

Chapter 3

Lyapunov stability

3.1 Trajectories, equilibria, and comparison functions

Definition 26. A *trajectory* of the autonomous ODE $\dot{x} = f(x)$ is a globally defined map $\varphi : \mathbb{R} \to \mathbb{R}^n$ satisfying

$$\dot{\varphi}(t) = f(\varphi(t)) \qquad \text{for every } t \in \mathbb{R}.$$

Definition 27. A point $x_{\mathrm{eq}} \in \mathbb{R}^n$ is an *equilibrium* of $\dot{x} = f(x)$ when $f(x_{\mathrm{eq}}) = 0$.

Definition 28. A *class* $\mathcal{K}$ function on $[0, a)$ is a continuous strictly increasing map $\alpha : [0, a) \to [0, b)$ with $\alpha(0) = 0$. The structure records both α and its inverse $\alpha^{-1} : [0, b) \to [0, a)$.

Definition 29. A *class* $\mathcal{K}_\infty$ function is a continuous strictly increasing map $\alpha : [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$.

3.2 Stability predicates

Definition 30. The equilibrium x_{eq} is *Lyapunov stable* when

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall \varphi,\ \mathrm{IsTrajectory}(\varphi, f) \ \Rightarrow\ \|\varphi(0) - x_{\mathrm{eq}}\| < \delta \ \Rightarrow\ \forall t \geq 0,\ \|\varphi(t) - x_{\mathrm{eq}}\| < \varepsilon.$$

Definition 31. The equilibrium x_{eq} is *locally asymptotically stable* (LAS) when it is Lyapunov stable (theorem 30) and there exists $c > 0$ such that every trajectory φ with $\|\varphi(0) - x_{\mathrm{eq}}\| < c$ satisfies $\varphi(t) \to x_{\mathrm{eq}}$ as $t \to \infty$.

Definition 32. The equilibrium x_{eq} is *globally asymptotically stable* (GAS) when it is Lyapunov stable (theorem 30) and every trajectory φ satisfies $\varphi(t) \to x_{\mathrm{eq}}$ as $t \to \infty$.

3.3 Sublevel sets and positive invariance

Definition 33. The *sublevel set* of $V : \mathbb{R}^n \to \mathbb{R}$ at level $c \in \mathbb{R}$ is

$$\Omega_c(V) \ = \ \{\, x \in \mathbb{R}^n \ : \ V(x) \leq c \,\}.$$

Definition 34. A set $S \subseteq \mathbb{R}^n$ is *positively invariant* for $\dot{x} = f(x)$ when every trajectory φ starting in S remains in S for all future time:

$$\varphi(0) \in S \;\Rightarrow\; \varphi(t) \in S \quad \forall\, t \geq 0.$$

Theorem 35. *If $V : \mathbb{R}^n \to \mathbb{R}$ is continuous and radially unbounded ($V(x) \to \infty$ as $\|x\| \to \infty$), then every sublevel set $\Omega_c(V)$ (theorem 33) is compact.*

Proof. Closed: $\Omega_c(V) = V^{-1}((-\infty, c])$ by continuity. Bounded: coercivity yields R with $\Omega_c(V) \subseteq \overline{B}(0, R)$. Compact: Heine–Borel in $\mathbb{R}^n$. □

3.4 Lyapunov functions

Definition 36. A function $V : \mathbb{R}^n \to \mathbb{R}$ is a *local Lyapunov function* on an open domain $D \ni x_{\mathrm{eq}}$ when V is smooth, $V(x_{\mathrm{eq}}) = 0$, $V > 0$ on $D \setminus \{x_{\mathrm{eq}}\}$, and the Lie derivative satisfies $\dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0$ for all $x \in D$.

Definition 37. A *strict local Lyapunov function* on D strengthens theorem 36: the Lie derivative satisfies $\dot{V}(x) < 0$ for all $x \in D \setminus \{x_{\mathrm{eq}}\}$, and there exists $c > 0$ such that the compact sublevel set $\Omega_c(V) \subseteq D$ (theorem 33).

Definition 38. A *global strict Lyapunov function* for $\dot{x} = f(x)$ at x_{eq} is a C^1 map $V : \mathbb{R}^n \to \mathbb{R}$ with $V(x_{\mathrm{eq}}) = 0$, $V > 0$ everywhere else, $\dot{V}(x) < 0$ on $\mathbb{R}^n \setminus \{x_{\mathrm{eq}}\}$, and all sublevel sets $\Omega_c(V)$ compact (coercivity).

Definition 39. The classical GAS Lyapunov certificate: a C^1 map $V : \mathbb{R}^n \to \mathbb{R}$ with $V(x_{\mathrm{eq}}) = 0$, $V > 0$ elsewhere, $\dot{V} < 0$ on $\mathbb{R}^n \setminus \{x_{\mathrm{eq}}\}$, and radially unbounded ($V(x) \to \infty$ as $\|x\| \to \infty$). Implies theorem 38 via theorem 35.

3.5 Milestone theorems for autonomous systems

Theorem 40. ***Lyapunov's stability theorem.*** *If V is a local Lyapunov function (theorem 36) for $\dot{x} = f(x)$ on a domain $D \ni x_{\mathrm{eq}}$, then x_{eq} is Lyapunov stable (theorem 30).*

Proof. Pick ε_0 so $\overline{B}(x_{\mathrm{eq}}, \varepsilon_0) \subseteq D$. Let $m = \min_{S_{\varepsilon'}} V > 0$. Choose δ with $V < m$ on $B(x_{\mathrm{eq}}, \delta)$. If $\|\varphi(t^*) - x_{\mathrm{eq}}\| \geq \varepsilon$, monotonicity of V gives $V(\varphi(t^*)) \leq V(\varphi(0)) < m \leq V(\varphi(t^*))$, a contradiction. □

Theorem 41. ***Lyapunov's local asymptotic stability theorem.*** *If V is a strict local Lyapunov function (theorem 37) and f is continuous, then x_{eq} is locally asymptotically stable (theorem 31).*

Proof. The compact sublevel set $\Omega_{c_0} \subseteq D$ is positively invariant. $V(\varphi(t)) \to L \geq 0$ by monotone convergence; $L = 0$ by compactness; $\varphi(t) \to x_{\mathrm{eq}}$. □

Theorem 42. ***Lyapunov's global asymptotic stability theorem.*** *If V is a global strict Lyapunov function (theorem 38) and f is continuous, then x_{eq} is globally asymptotically stable (theorem 32).*

Proof. Stability from theorem 40. For convergence: $V(\varphi(t)) \to L \geq 0$ by monotone convergence; $L = 0$ by a LaSalle-type argument; $\varphi(t) \to x_{\mathrm{eq}}$ by coercivity. □

Theorem 43. ***Corollary.*** *If V is a radially unbounded strict Lyapunov function (theorem 39) and f is continuous, then x_{eq} is globally asymptotically stable.*

Proof. Radial unboundedness gives compact sublevel sets (theorem 35), so V satisfies theorem 38; apply theorem 42. □

3.6 LaSalle's invariance principle

Theorem 44. ***LaSalle's invariance principle.*** *Let Ω be compact and positively invariant for $\dot{x} = f(x)$, $V \in C^1$ with $\dot{V}(x) \le 0$ on Ω, and $M \subseteq \Omega$ closed. If the ω-limit set of any trajectory φ starting in Ω satisfies $\omega(\varphi) \subseteq M$, then $\varphi(t) \to M$ as $t \to \infty$.*

Proof. □

Theorem 45. ***Barbashin's theorem.*** *If $V \in C^1$ is positive definite on D, $\dot{V} \le 0$ on D, there exists a compact sublevel set $\Omega_c \subseteq D$, and every trajectory starting in Ω_c has ω-limit set $\subseteq \{x_{\mathrm{eq}}\}$, then x_{eq} is locally asymptotically stable (theorem 31).*

Proof. □

Theorem 46. ***Krasovskii's theorem.*** *If $V \in C^1$ is positive definite, $\dot{V} \le 0$ on $\mathbb{R}^n$, radially unbounded, and every trajectory has ω-limit set $\subseteq \{x_{\mathrm{eq}}\}$, then x_{eq} is globally asymptotically stable (theorem 32).*

Proof. □

3.7 Lyapunov class $\mathcal{K}$ bounds

Theorem 47. ***Class K sandwich bounds.*** *For any continuous positive-definite function $V : \mathbb{R}^n \to \mathbb{R}$ on $\overline{B}(0, r)$, there exist class $\mathcal{K}$ functions α_1, α_2 such that*

$$\alpha_1(\|x\|) \;\le\; V(x) \;\le\; \alpha_2(\|x\|) \qquad \forall\, \|x\| \le r.$$

Proof. Construct α_1 from the infimum of V on annuli (lower bound), and α_2 from the supremum of V on balls (upper bound). Each is sandwiched by a class $\mathcal{K}$ function via the axioms in `LeanForControl.axioms`. □

3.8 Class $\mathcal{KL}$ functions and the Osgood construction

Definition 48. A *class $\mathcal{KL}$* function on $[0, a) \times [0, \infty)$ is continuous, class $\mathcal{K}$ in the first argument, and for each fixed $r > 0$ is strictly decreasing and tends to 0 as $s \to \infty$. It arises as the bound $\|x(t)\| \le \beta(\|x_0\|, t)$ in asymptotic stability estimates.

Theorem 49. ***Osgood construction.*** *Given a class $\mathcal{K}$ function α satisfying the Osgood condition $\int_0^{\varepsilon} \frac{1}{\alpha(x)}\, dx = +\infty$, the function $\sigma(r, s) = \eta^{-1}(\eta(r) + s)$, where $\eta(y) = -\int_{\mathrm{base}}^{y} \frac{1}{\alpha(x)}\, dx$, is a class $\mathcal{KL}$ function (theorem 48).*

Proof. □